RELATIVITY AND COSMOLOGY I

Problem Set 2 — Fall 2023

1. Index Gymnastics part 2

In part two of this exercise, we consider more complicated properties of tensors

(a) Lorentz transformations Λ can be defined as the transformations which leave the metric of Minkowski space η invariant:

$$\Lambda^{\mu}{}_{\nu}\eta^{\nu\tau}\Lambda^{\sigma}{}_{\tau} = \eta^{\mu\sigma} \,. \tag{1}$$

Use this to prove that the inverse of a Lorentz transformation $\Lambda^{\mu}{}_{\nu}$ is $(\Lambda^{-1})^{\nu}{}_{\mu} = \Lambda_{\mu}{}^{\nu}$.[1]

(b) Under a Lorentz transformation, tensor fields transform as follows

$$T^{\mu'_1\cdots\mu'_n}{}_{\nu'_1\cdots\nu'_m}(x') = \Lambda^{\mu'_1}{}_{\mu_1}\cdots\Lambda^{\mu'_n}{}_{\mu_n}\Lambda_{\nu'_1}{}^{\nu_1}\cdots\Lambda_{\nu'_m}{}^{\nu_m}T^{\mu_1\cdots\mu_n}{}_{\nu_1\cdots\nu_m}(x) \,. \tag{2}$$

Prove that the product of two tensor fields of rank s and t transforms as a tensor field of rank $(s+t)$.

(c) Completely symmetric and antisymmetric tensors satisfy $T_{\mu\nu} = \pm T_{\nu\mu}$, where the plus sign stands for the symmetric and the minus sign for the antisymmetric one. Show that, if $S_{\mu\nu}$ is a symmetric tensor and $A_{\mu\nu}$ is an antisymmetric tensor, then $S_{\mu\nu}A^{\mu\nu} = 0$.

(d) Show that, if $S_{\mu\nu}$ is symmetric and $B_{\mu\nu}$ is arbitrary, $S_{\mu\nu}B^{\mu\nu} = \frac{1}{2}S_{\mu\nu}\left(B^{\mu\nu} + B^{\nu\mu}\right)$.

(e) Show that, if $A_{\mu\nu}$ is antisymmetric and $B_{\mu\nu}$ is arbitrary, $A_{\mu\nu}B^{\mu\nu} = \frac{1}{2}A_{\mu\nu}\left(B^{\mu\nu} - B^{\nu\mu}\right)$.

(f) How many independent components does an arbitrary rank-r tensor in a n-dimensional spacetime have?

(g) How many independent components does a rank-r tensor that is symmetric in s of its indices in a n-dimensional spacetime have?

(h) How many independent components does a rank-r tensor that is antisymmetric on a of its indices in a n-dimensional spacetime have? What happens when $a > n$?

(i) How many indipendent components does a rank-4 tensor that is totally antisymmetric in its indices in a n-dimensional spacetime? Comment the result and how it relates to Problem 2 of this set.

[1] The definition of $\Lambda_{\mu}{}^{\nu}$ is $\Lambda_{\mu}{}^{\nu} = \eta_{\mu\sigma}\eta^{\nu\tau}\Lambda^{\sigma}{}_{\tau}$.

2. The Levi-Civita Symbol

The Levi-Civita symbol $\tilde{\epsilon}_{\kappa\lambda\mu\nu}$, also called the **totally antisymmetric symbol**, is defined through

$$\tilde{\epsilon}_{\kappa\lambda\mu\nu} = \begin{cases} +1 & \text{if } \kappa\lambda\mu\nu \text{ is an even permutation of } 0123 \;, \\ -1 & \text{if } \kappa\lambda\mu\nu \text{ is an odd permutation of } 0123 \;, \\ \;\;0 & \text{otherwise} \;. \end{cases} \tag{3}$$

We use the tilde ($\sim$) to distinguish the Levi-Civita *symbol* from the Levi-Civita *tensor*, which we will introduce in Problem Set 4.

(a) Show that since $\tilde{\epsilon}_{0123} = 1$, then $\tilde{\epsilon}^{0123} = -1$.
Hint: keep in mind that indices are raised and lowered in flat space through the Minkowski metric.

In general, contractions of the totally antisymmetric tensor can be expressed in terms of products of delta functions as

$$\tilde{\epsilon}_{\alpha_1\alpha_2\ldots\alpha_m\mu_1\mu_2\ldots\mu_n}\tilde{\epsilon}^{\alpha_1\alpha_2\ldots\alpha_m\nu_1\nu_2\ldots\nu_n} = (-1)^s m!\, n!\, \delta^{[\nu_1}_{[\mu_1} \cdots \delta^{\nu_n]}_{\mu_n]} \;, \tag{4}$$

where s is the number of negative eigenvalues of the metric.

(b) Use (4) to show that $\tilde{\epsilon}_{\alpha\beta\gamma\delta}\tilde{\epsilon}^{\kappa\lambda\gamma\delta} = -2(\delta^\kappa_\alpha\delta^\lambda_\beta - \delta^\lambda_\alpha\delta^\kappa_\beta)$.

(c) Prove that $\tilde{\epsilon}_{\kappa\lambda\mu\nu}\tilde{\epsilon}^{\kappa\lambda\mu\nu} = -4!$.

(d) Argue that $\tilde{\epsilon}_{\kappa\lambda\mu\nu}M^\kappa{}_\alpha M^\lambda{}_\beta M^\mu{}_\gamma M^\nu{}_\delta = \det(M)\,\tilde{\epsilon}_{\alpha\beta\gamma\delta}$, for any 2 index tensor M .

3. Electromagnetism with Tensors

In electromagnetism, the potential four-vector is defined through $A^\mu = (\phi, \vec{A})$. The electric and magnetic fields are related to the potential via

$$\vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t} \,, \qquad \vec{B} = \vec{\nabla} \times \vec{A} \,. \tag{5}$$

(a) Compute all the entries of

$$F^{\mu\nu} \equiv 2\partial^{[\mu}A^{\nu]} = \partial^\mu A^\nu - \partial^\nu A^\mu \tag{6}$$

when represented in matrix form and express them in terms of components of $\vec{E}$ and $\vec{B}$ only. This is called the **field strength tensor**.

(b) Show that

$$\partial_\mu F^{\nu\mu} = J^\nu \,, \qquad \tilde{\epsilon}^{\mu\nu\rho\sigma}\partial_\mu F_{\rho\sigma} = 0 \,, \tag{7}$$

encode all of Maxwell's equations. We used $J^\mu \equiv (\rho, \vec{J})$.

(c) Write down the transformation law for the field strength tensor under a Lorentz transformation.

(d) Specialize to a boost along the x direction:[2]

$$\Lambda = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \tag{8}$$

Find the transformation laws of the $\vec{E}$ and $\vec{B}$ field components under such a boost.

4. The Energy-Momentum Tensor

The energy-momentum tensor (also called the **stress-energy tensor**) encodes the flux of energy and momentum through spacetime. For electromagnetism, it is given by

$$T^{\mu\nu} = F^{\mu\lambda} F^{\nu}{}_{\lambda} - \frac{1}{4}\eta^{\mu\nu} F^{\lambda\sigma} F_{\lambda\sigma} . \tag{9}$$

(a) Compute its trace $T^{\mu}{}_{\mu} = \eta_{\mu\nu} T^{\mu\nu}$ in general dimension n. Do you see that something special happens in $n = 4$ spacetime dimensions?

(b) Verify that

$$T^{00} = \rho = \frac{1}{2}\vec{E}^2 + \frac{1}{2}\vec{B}^2 \tag{10}$$

is the electromagnetic energy density.

(c) Verify that

$$T^{0i} = \vec{S}_i , \tag{11}$$

where $\vec{S}$ is the Poynting vector. Notice that T^{0i} is also the momentum density (component i) in the electromagnetic field.

(d) Using Maxwell's equations (7) show that

$$\partial_\mu T^{\mu\nu} = J_\alpha F^{\alpha\nu} . \tag{12}$$

Rewrite the $\nu = 0$ component of this equation using the relations (10) and (11). What is the physical meaning of this equation? What about the spatial components $\nu = 1, 2, 3$?

[2] Can you tell that this is exactly the same matrix we wrote down in problem set 1?